Deriving "τ_p = 63.2% of Process Step Response" Rule

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To derive the "time constant, τ_P = 63.2% of Process Step Response" rule used in model fitting, we must solve a linear first order ordinary differential equation (ODE):

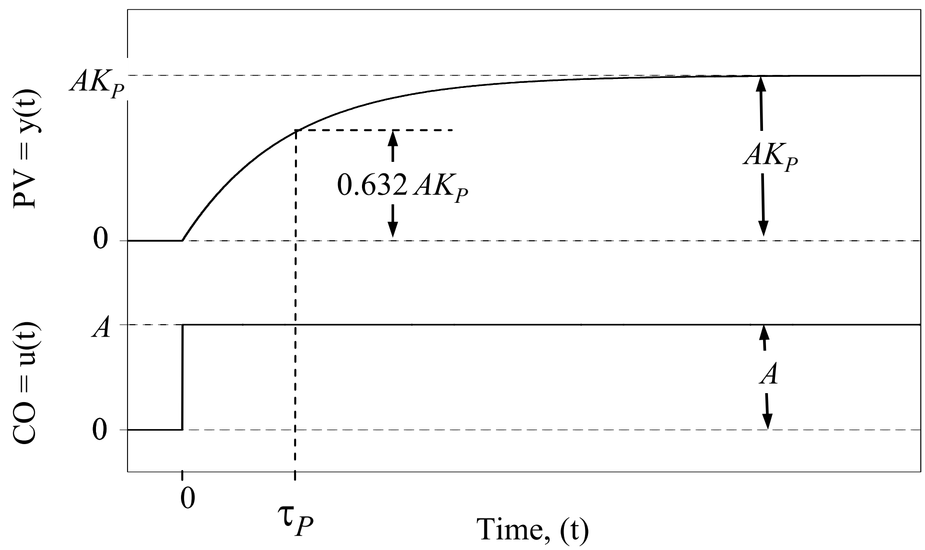

Figure 1 – Response of true first order process to a step change in controller output

The figure above shows the open loop step response of a true first order process model:

$$\tau_P \frac{dy(t)}{dt} + y(t) = K_P\, u(t) \qquad \text{where} \qquad y(0) = 0$$

As shown in Fig 1, the measured process variable, $y(t)$, and controller output signal, $u(t)$, are initially at steady state with $y(t) = u(t) = 0$ for $t < 0$. Other parameters include the process time constant, τ_P, and process gain, K_P.

At time $t = 0$, the controller output is stepped to $u(t) = A$, where it remains for the duration of the experiment. Hence, the first order model becomes

$$\tau_P \frac{dy(t)}{dt} + y(t) = AK_P$$

To solve this ODE, rearrange as:

$$\frac{dy(t)}{dt} + \frac{1}{\tau_P} y(t) = \frac{AK_P}{\tau_P}$$

First compute the integrating factor, $\mu = e^{\int (1/\tau_p)dt} = e^{t/\tau_p}$. Solving the ODE using this integrating factor yields

$$y(t) = \frac{1}{e^{t/\tau_P}} \left[\int e^{t/\tau_P} \left(\frac{AK_P}{\tau_P} \right) dt + c_1 \right]$$

$$= e^{-t/\tau_P} \left[\frac{AK_P}{\tau_P} \int e^{t/\tau_P} dt + c_1 \right]$$

Recall that $\int e^{ax} dx = \frac{1}{a} e^{ax}$

and thus $\int e^{t/\tau_P} dt = \tau_P e^{t/\tau_P}$

so $y(t) = e^{-t/\tau_P} \left[AK_P e^{t/\tau_P} + c_1 \right]$

$$= AK_P + c_1 e^{-t/\tau_P}$$

Next, apply the initial condition: @ $t = 0$, $y = 0$, $0 = AK_P + c_1$, and thus $c_1 = -AK_P$.

Substituting and rearranging, we obtain the solution to the ODE:

$$\boxed{y(t) = AK_P \left[1 - e^{-t/\tau_P} \right]}$$

After the passage of one time constant, time $t = \tau_P$, the solution becomes

$$y(\tau_P) = AK_P \left[1 - e^{-\tau_P/\tau_P} \right] = AK_P \left[1 - e^{-1} \right]$$

Therefore, the measured process variable step response at time $t = \tau_P$ is

$$\boxed{y(\tau_P) = 0.632 AK_P}$$

As we set out to show, at time t equals one time constant, τ_P, the measured process variable, y(t), will have traveled to 63.2% of the total change that it will ultimately experience.